

Master Numbers: Generalized Parametric Factorials

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Abstract

We introduce **master numbers** (generalized parametric factorials), a parametric family of products that generalize combinatorial sequences such as factorials, binomial coefficients, and rising/falling factorials. The master numbers are defined by the product formula:

$$mstr(m; s; r; t) = {}_r^m t^s = \prod_{j=0}^{t-1} (m + st - rj)$$

where m , s , r are parameters and t is the independent argument. This framework unifies diverse combinatorial constructs under a single parametric representation, providing powerful algebraic tools for series expansions and equation solving. Detailed applications will be addressed in subsequent work.

Keywords: Master numbers, generalized factorials, combinatorial sequences, parametric products.

1 Introduction

Master numbers extend classical combinatorial concepts by introducing adjustable parameters m , s , r , which control the structure of the product. For example:

When $m = 1$, $s = 1$, $r = 1$, the master number reduces to the standard factorial: ${}_1^1 t^1 = t!$.

Other parameter choices yield falling factorials, binomial coefficients, and power functions (see Table 1).

Interactive Lab: The computational implementation of master numbers is available in the online laboratory: [Zenodo:10.5281/zenodo.15633781](https://zenodo.org/doi/10.5281/zenodo.15633781).

2 Definition and Properties

2.1 Master Number Formula

Building upon the foundational work of [3] (Theorem on Lightning Inversion of Equations), we define the master number for $t \in \mathbb{N}, t \geq 1$:

$${}^m_1s_t = \prod_{\substack{j < t \\ j > 0}} (m + st - rj)$$

For the base case $t = 1$, the product is empty (no terms to multiply), and by convention, we define:

$${}^m_1s_1 = 1$$

This aligns with the combinatorial interpretation where an empty product equals the multiplicative identity (1), analogous to $0! = 1$.

Note: The case $t = 0$ is not considered in this framework, as it would require a separate definition depending on the context (e.g., extending to $t \in \mathbb{Z}_{>0}$ could involve setting ${}^m_1s_0 = 1$ for consistency, but this lies beyond the scope of the current work).

For $t \geq 2$, the explicit form is:

$${}^m_1s_t = (m + st - r)(m + st - 2r) \cdots (m + st - r(t - 1))$$

Key Parameters:

- m : Base offset.
- s : Scaling factor for the term st .
- r : Scaling factor for the step j .
- t : Independent argument (e.g., series index).

2.2 Special Cases

Table 1: Examples of Master Numbers

Case	General Form	Example
Gamma function $\Gamma(t)$	0_11_t	${}^0_11_3 = \Gamma(3) = 2 \times 1$
Factorial $t!$	1_11_t	${}^1_11_3 = 3! = 3 \times 2$
r-fold factorial of n $n!_{(r)}$	${}^{n-rt}_r1_{r,t+1}$	$t = \lceil \frac{n}{r} \rceil$
Falling factorial $(n)_k$	${}^{n+1}_11^0_{k+1}$	$(5)_3 = {}^6_11^0_4 = 5 \times 4 \times 3$
Binomial coefficient $\binom{n}{k}$	$\frac{{}^{n+1}_11^0_{k+1}}{k!}$	$\binom{4}{2} = \frac{{}^5_11^0_3}{2!} = 6$
Rising factorial $n^{(k)}$	${}^{n-1}_-11^0_{k+1}$	$3^{(3)} = {}^2_11^0_4 = 3 \times 4 \times 5$
Power functions m^{t-1}	${}^m_01^0_t$	${}^5_01^0_3 = (5)(5)$
Derivative of power $(x^t)^{(t)}$	${}^1_11^1_t$	$(x^t)^{(t)} = {}^1_11^1_t = t!$

2.3 Examples

2.3.1 r-fold factorial of n

$$n!_{(r)} = {}^{n-rt}r!_{t+1}, \quad t = \left\lceil \frac{n}{r} \right\rceil$$

$$\begin{aligned} 5!!! &= 5!_{(3)} = {}^{5-3t}3!_{t+1}, \quad t = \left\lceil \frac{5}{3} \right\rceil = 2 \\ &= {}^{-1}3!_3 = (-1 + 9 - 3)(-1 + 9 - 6) = 5 \times 2 \end{aligned}$$

$$\begin{aligned} 6!!! &= 6!_{(3)} = {}^{6-3t}3!_{t+1}, \quad t = \left\lceil \frac{6}{3} \right\rceil = 2 \\ &= {}^0 3!_3 = (9 - 3)(9 - 6) = 6 \times 3 \end{aligned}$$

$$\begin{aligned} 7!!! &= 7!_{(3)} = {}^{7-3t}3!_{t+1}, \quad t = \left\lceil \frac{7}{3} \right\rceil = 3 \\ &= {}^{-2}3!_4 = (-2 + 12 - 3)(-2 + 12 - 6)(-2 + 12 - 9) = 7 \times 4 \times 1 \end{aligned}$$

2.3.2 Falling factorial

$$(n)_k = {}^{n+1}1!_{k+1}^0$$

$$(5)_3 = {}^6 1!_4^0 = (6 - 1)(6 - 2)(6 - 3) = 5 \times 4 \times 3$$

2.3.3 Binomial coefficient

$$\binom{n}{k} = \frac{{}^{n+1}1!_{k+1}^0}{k!}$$

$$\binom{4}{2} = \frac{{}^5 1!_3^0}{2!} = \frac{(5 - 1)(5 - 2)}{2} = 6$$

2.3.4 Rising factorial

$$n^{(k)} = {}^{n-1}1!_{k+1}^0$$

$$3^{(3)} = {}^2 1!_4^0 = (2 + 1)(2 + 2)(2 + 3) = 3 \times 4 \times 5$$

2.3.5 Power functions

$${}^m 1!_t^0 = m^{t-1}$$

$${}^m 1!_t^s = (m + st)^{t-1}$$

$${}^5 1!_3^0 = (5)(5)$$

$${}^5 1!_3^4 = (5 + 12)(5 + 12)$$

2.4 Key Identities

Duality:

$${}_{-r}^m!_t^0 = (m+r)(m+2r)\cdots(m+r(t-2))(m+r(t-1))$$

$${}_{r}^m!_t^r = (m+r(t-1))(m+r(t-2))\cdots(m+2r)(m+r)$$

Derivation:

$$mstr(m; 0; -r; t) = mstr(m; r; r; t)$$

$${}_{-r}^m!_t^0 = {}_{r}^m!_t^r$$

Example:

$${}_{-2}^1!_4^0 = (1+2)(1+4)(1+6) = 3 \times 5 \times 7$$

$${}_{2}^1!_4^2 = (1+8-2)(1+8-4)(1+8-6) = 7 \times 5 \times 3$$

2.5 Merge Operation (@)

The merge operation, originally developed for composite functions in [2] (Natural and composite functions), extends naturally to master numbers.

Master numbers can be merged into composite structures:

$${}_{r}^m!_{t_1, t_2, \dots}^{s_1, s_2, \dots} = \prod_{j=1}^{j < t_1 + t_2 + \dots} (m + s_1 t_1 + s_2 t_2 + \dots - rj)$$

$${}_{r}^m!_{t_1, t_2, t_3}^{s_1, s_2, s_3} = \prod_{j=1}^{t_1 + t_2 + t_3 - 1} (m + s_1 t_1 + s_2 t_2 + s_3 t_3 - rj)$$

This operation allows only paired swaps of indices (e.g., (s_1, t_1) with (s_2, t_2)) and generalizes multi-indexed series expansions.

Special Case for $t = 1$: When merging master numbers with $t = 1$, the resulting monolith is no longer equal to 1, despite the individual master numbers ${}_{r}^m!_1^s = 1$. The merged structure becomes:

$${}_{r}^m!_1^{s_1} @ {}_{r}^m!_1^{s_2} = (m + s_1 + s_2 - r)$$

For three merged master numbers with $t = 1$:

$${}_{r}^m!_1^{s_1} @ {}_{r}^m!_1^{s_2} @ {}_{r}^m!_1^{s_3} = (m + s_1 + s_2 + s_3 - r)(m + s_1 + s_2 + s_3 - 2r)$$

This demonstrates that the merge operation creates fundamentally new structures that cannot be obtained through simple multiplication of the original master numbers.

3 Applications

3.1 Series Inversion

Following the universal methods framework established in [1] (Universal function and universal methods for eliminating unknowns), we construct the universal function

$${}_{r \cdot x}^{m, s} = m + \sum_{t=1}^{\infty} \left(\frac{x^t}{t!} \times {}_{r \cdot t}^{m, s} \right)$$

Master numbers enable efficient inversion of equations of the form:

$$y = v e^{x_1 y^{s_1} + x_2 y^{s_2} + \dots} = v \times {}_{0 \cdot x_1 v^{s_1}, x_2 v^{s_2}, \dots}^{1, s_1, s_2, \dots}$$

and

$$p y^r = q + x_1 y^{s_1} + x_2 y^{s_2} + \dots = q \times {}_{1 \cdot \frac{x_1 (q/p)^{s_1/r}, x_2 (q/p)^{s_2/r}, \dots}^{\frac{1, s_1/r, s_2/r, \dots}}{q}}$$

by unifying component-specific series into a single master series.

From two equations

$$y = (1 + r x y^s)^{\frac{1}{r}} = {}_{r \cdot x}^{1, s}, \quad r \neq 0$$

$$y = e^{x y^s} = {}_{0 \cdot x}^{1, s}$$

we can derive 8 equations using the following steps:

1. Set $s = 0$ in both the equation and the master number.
2. Substitute $y = e^u$ and take the logarithm of the equation.

$$y = e^{x y^s} = {}_{0 \cdot x}^{1, s}$$

$$\ln {}_{r \cdot x}^{1, s} = {}_{r \cdot x}^{0, s}$$

$$u = x e^{s u} = {}_{0 \cdot x}^{0, s}$$

3.2 Multinomial Generalizations

The merge operation (@) naturally extends binomial/multinomial coefficients

$$(1 + x_1 + x_2 + \dots)^r = {}_{1/r \cdot r x_1, r x_2, \dots}^{1, 0, 0, \dots}, \quad r \in \mathbb{N}$$

$$\left(1 + \frac{x}{2}\right)^2 = {}_{1/2 \cdot x}^{1, 0} = 1 + {}_{1/2 \cdot 1}^{1, 0} \frac{x}{1!} + {}_{1/2 \cdot 2}^{1, 0} \frac{x^2}{2!}$$

$${}_{1/2 \cdot 3}^{1, 0} = \prod_{j=1}^{3-1} (1 - j/2) = (1 - 1/2)(1 - 2/2) = 0, \quad {}_{1/2 \cdot 4}^{1, 0} = 0, \quad {}_{1/2 \cdot 5}^{1, 0} = 0, \dots$$

$$(1 + x_1/2 + x_2/2)^2 = {}_{1/2 \cdot x_1, x_2}^{1,0,0}$$

$$\begin{aligned} {}_{1/2 \cdot x_1, x_2}^{1,0,0} &= \left(1 + {}_{1/2 \cdot 1}^{1_1 0} \frac{x_1^1}{1!} + {}_{1/2 \cdot 2}^{1_1 0} \frac{x_1^2}{2!}\right) @ \left(1 + {}_{1/2 \cdot 1}^{1_2 0} \frac{x_2^1}{1!} + {}_{1/2 \cdot 2}^{1_2 0} \frac{x_2^2}{2!}\right) \\ &= 1 + {}_{1/2 \cdot 1}^{1_1 0} \frac{x_1^1}{1!} + {}_{1/2 \cdot 2}^{1_1 0} \frac{x_1^2}{2!} + {}_{1/2 \cdot 1}^{1_2 0} \frac{x_2^1}{1!} + {}_{1/2 \cdot 1, 1}^{1_1 0, 0} \frac{x_2^1}{1!} \frac{x_1^1}{1!} + {}_{1/2 \cdot 2}^{1_2 0} \frac{x_2^2}{2!} \\ &= 1 + \frac{x_1^1}{1!} + \frac{x_1^2}{4} + \frac{x_2^1}{1!} + \frac{x_1^1 x_2^1}{2} + \frac{x_2^2}{4} \end{aligned}$$

$${}_{1/2 \cdot 2}^{1_1 0} @ {}_{1/2 \cdot 1}^{1_1 0} = {}_{1/2 \cdot 2, 1}^{1_1 0, 0} = \prod_{j=1}^{2+1-1} (1 - j/2) = (1 - 1/2)(1 - 2/2) = 0$$

$${}_{1/2 \cdot 2}^{1_2 0} @ {}_{1/2 \cdot 2}^{1_2 0} = {}_{1/2 \cdot 2, 2}^{1_2 0, 0} = 0$$

Thus, multinomial coefficients can be naturally expressed within the master number framework.

4 Conclusion

Master numbers provide a unified framework for combinatorial sequences, offering:

- **Generality:** Captures classical and exotic sequences via parameter tuning.
- **Algebraic Flexibility:** The merge operation (@) supports complex series manipulations.
- **Computational Efficiency:** Simple recursive implementation.

Future Work: Comprehensive examples of series expansions and equation solving will be presented in a follow-up paper.

JavaScript Implementation

```
function mstr(m, s, r, t) {
  let u = 1;
  for (let j = 1; j < t; j++) {
    u *= (m + s * t - r * j);
  }
  return u;
}
```

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Code & Interactive Examples: <http://glax-plato.ru/>.

Zenodo: Master-J Laboratory.

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